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Reliability-based sensitivity of mechanical components with arbitrary distribution parameters †

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Abstract

This paper presents a reliability-based sensitivity method for mechanical components with arbitrary distribution parameters. Techniques from the perturbation method, the Edgeworth series, the reliability-based design theory, and the sensitivity analysis approach were employed directly to calculate the reliability-based sensitivity of mechanical components on the condition that the first four moments of the original random variables are known. The reliability-based sensitivity information of the mechanical components can be accurately and quickly obtained using a practical computer program. The effects of the design parameters on the reliability of mechanical components were studied. The method presented in this paper provides the theoretic basis for the reliability-based design of mechanical components.

Keywords: Mechanical components; Reliability-based sensitivity; Probabilistic perturbation method; Arbitrary distribution parameters; The Edgeworth series

1. Introduction

In recent years, efforts have been made to develop a means of quantifying uncertainties, and their combined effect on reliability, in engineering systems. Theoretically, these uncertainties are modeled as random variables governed by joint probability density or distribution functions. In actual engineering design, the exact joint probability density functions are often unavailable or difficult to obtain owing to insufficient data. Not infrequently, the available data is sufficient only to evaluate the first few moments, including the mean, variance and correlations.

Traditionally, engineering system reliability is achieved through the use of coefficients of safety and the adoption of conservative assumptions in the design process. The traditional approach, as it lacks the logical basis for addressing uncertainties, cannot quantitatively assess the level of safety or reliability. Assuring the performance of new systems for which there is no established basis of calibration, obviously would be problematic. Over the last three decades, diverse design methods have been developed in engineering design to ensure the reliability of product systems [1–6].

A great number of reliability-based design approaches have been formulated on the assumption that original random variables are normal distributions. When non-normal original random variables are involved, Rosenblatt transformation [7] and the Hasofer Lind-Rackwitz Fiessler method [8] are often used. However, as already noted, the exact joint probability density function is often unavailable or difficult to obtain owing to insufficient data, which might be sufficient only to evaluate the first few moments. In this situation of incomplete information, it is difficult to employ the Rosenblatt transformation or the Hasofer Lind-Rackwitz Fiessler method and to obtain the reliability-sensitivity design parameters. Thus, an alternative computational method of reliability-sensitivity design with arbitrary distribution parameters is required.

Reliability-based sensitivity analysis has been widely applied in reliability engineering design to estimate the effect of a change in a random variable on the probability of structural failure, to obtain valuable information on model behavior, to evaluate the accuracy of a model, and other purposes. Structural reliability sensitivity calculation methods have been successfully developed, for both efficiency and accuracy [9–10]. These include, among others, an efficient adaptive importance sampling (AIS) method for component and system reliability-based sensitivities [11], an approximate solution technique for

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reliability-based sensitivity of non-linear random vibration rotor-stator systems with rubbing [12], and a reliability-based sensitivity analysis protocol based on an efficient simulation approach [13].

The present study focused on an extension of reliability-based sensitivity analysis of mechanical components with arbitrary distribution parameters. This paper proposes a numerical approach to calculation of the reliability-based sensitivity of mechanical components, incorporating the perturbation method, the reliability theory and sensitivity analysis. This method can be used to obtain the reliability-based sensitivity information of mechanical components accurately and quickly. The results of numerical computation demonstrate that the method presented is a convenient and practical reliability-based sensitivity design approach.

2. Perturbation method of reliability design

A fundamental problem in reliability analysis is computation of the multi-fold integral of the reliability R

$$R=\int_{g(\mathbf{X})>0} f_{\mathrm{X}}(\mathbf{X})\mathrm{d}\mathbf{X} \tag{1}$$

in which $f_{\boldsymbol{X}}(\boldsymbol{X})$ denotes the probability density function of a random parameter vector $\boldsymbol{X}=(X_1\ X_2\ \cdots\ X_n)^{\mathrm{T}}$, and $g(\boldsymbol{X})$ defines the state function representing the safe state or failure state

$$\left.\begin{aligned} &g(\mathbf{X})\le 0 \quad \text{failure state}\\ &g(\mathbf{X})>0 \quad \text{safe state}\end{aligned}\right\} \tag{2}$$

where $g(\boldsymbol{X})=0$ is the limit-state equation representing an n-dimensional surface that may be called the “limit-state surface” or “failure surface”.

The vector of random parameters $\boldsymbol{X}$ and the state function $g(\boldsymbol{X})$ are expanded as

$$\mathbf{X}=\mathbf{X}_{\mathrm{d}}+\varepsilon\mathbf{X}_{\mathrm{p}} \tag{3}$$

$$g(\mathbf{X})=g_{\mathrm{d}}(\mathbf{X})+\varepsilon g_{\mathrm{p}}(\mathbf{X}) \tag{4}$$

where ε is a small parameter. Subscript d in Eqs. (3) and (4) represents the certainty aspect of the random parameters, and subscript p, the random aspect, having a zero mean value in the random parameters. Obviously, it is necessary for the value of the random aspect to be smaller than the value of the certain aspect. Both sides of Eqs. (3) and (4) are evaluated for the mean value of random variables as follows:

$$\mathrm{E}(\mathbf{X})=\mathrm{E}(\mathbf{X}_{\mathrm{d}})+\varepsilon\mathrm{E}(\mathbf{X}_{\mathrm{p}})=\mathbf{X}_{\mathrm{d}}=\bar{\mathbf{X}} \tag{5}$$

$$\mu_{\mathrm{g}}=\mathrm{E}[g(\mathbf{X})]=\mathrm{E}[g_{\mathrm{d}}(\mathbf{X})]+\varepsilon\mathrm{E}[g_{\mathrm{p}}(\mathbf{X})]=g_{\mathrm{d}}(\mathbf{X})\approx g(\bar{\mathbf{X}}) \tag{6}$$

Similarly, according to Kronecker algebra [14], both sides of Eqs. (3) and (4) are evaluated for the variance, the third moment and the fourth moment of the random variables and the state function as follows:

$$\mathrm{Var}(\mathbf{X})=\mathrm{E}\left\{[\mathbf{X}-\mathrm{E}(\mathbf{X})]^{[2]}\right\}=\varepsilon^2\mathrm{E}\left[\mathbf{X}_{\mathrm{p}}^{[2]}\right] \tag{7a}$$

$$\mathbf{C}_3(\mathbf{X})=\mathrm{E}\left\{[\mathbf{X}-\mathrm{E}(\mathbf{X})]^{[3]}\right\}=\varepsilon^3\mathrm{E}\left[\mathbf{X}_{\mathrm{p}}^{[3]}\right] \tag{7b}$$

$$\mathbf{C}_4(\mathbf{X})=\mathrm{E}\left\{[\mathbf{X}-\mathrm{E}(\mathbf{X})]^{[4]}\right\}=\varepsilon^4\mathrm{E}\left[\mathbf{X}_{\mathrm{p}}^{[4]}\right] \tag{7c}$$

$$\mathrm{Var}[g(\mathbf{X})]=\mathrm{E}\left\{[g(\mathbf{X})-\mathrm{E}(g(\mathbf{X}))]^{[2]}\right\}=\varepsilon^2\mathrm{E}\left\{[g_{\mathrm{p}}(\mathbf{X})]^{[2]}\right\} \tag{8a}$$

$$\mathrm{C}_3[g(\mathbf{X})]=\mathrm{E}\left\{[g(\mathbf{X})-\mathrm{E}(g(\mathbf{X}))]^{[3]}\right\}=\varepsilon^3\mathrm{E}\left\{[g_{\mathrm{p}}(\mathbf{X})]^{[3]}\right\} \tag{8b}$$

$$\mathrm{C}_4[g(\mathbf{X})]=\mathrm{E}\left\{[g(\mathbf{X})-\mathrm{E}(g(\mathbf{X}))]^{[4]}\right\}=\varepsilon^4\mathrm{E}\left\{[g_{\mathrm{p}}(\mathbf{X})]^{[4]}\right\} \tag{8c}$$

where the Kronecker power is $\boldsymbol{P}^{[k]}=\boldsymbol{P}\otimes\boldsymbol{P}^{[k-1]}=\boldsymbol{P}\otimes\boldsymbol{P}\otimes\cdots\otimes\boldsymbol{P}$, and the symbol $\otimes$ represents the Kronecker product defined as $(\boldsymbol{A})_{p\times q}\otimes(\boldsymbol{B})_{s\times t}=[a_{ij}\boldsymbol{B}]_{ps\times qt}$.

By expanding the state function $g_{\mathrm{p}}(\boldsymbol{X})$ to a first-order approximation in a Taylor series of vector-valued functions and matrix-valued functions at a point $\mathrm{E}(\boldsymbol{X})=\boldsymbol{X}_{\mathrm{d}}$ on the failure surface $g_{\mathrm{p}}(\boldsymbol{X}_{\mathrm{d}})=0$, the expression of $g_{\mathrm{p}}(\boldsymbol{X})$ is given as

$$g_{\mathrm{p}}(\mathbf{X})=\frac{\partial g_{\mathrm{d}}(\mathbf{X})}{\partial\mathbf{X}^{\mathrm{T}}}\mathbf{X}_{\mathrm{p}} \tag{9}$$

Substituting Eq. (9) into Eqs. (8), we obtain

$$\sigma_{\mathrm{g}}^2=\mathrm{Var}[g(\mathbf{X})]=\varepsilon^2\mathrm{E}\left[\left(\frac{\partial g_{\mathrm{d}}(\mathbf{X})}{\partial\mathbf{X}^{\mathrm{T}}}\right)^{[2]}\mathbf{X}_{\mathrm{p}}^{[2]}\right]=\left(\frac{\partial g_{\mathrm{d}}(\mathbf{X})}{\partial\mathbf{X}^{\mathrm{T}}}\right)^{[2]}\mathrm{Var}(\mathbf{X}) \tag{10a}$$

$$\theta_{\mathrm{g}}=\mathrm{C}_3[g(\mathbf{X})]=\varepsilon^3\mathrm{E}\left[\left(\frac{\partial g_{\mathrm{d}}(\mathbf{X})}{\partial\mathbf{X}^{\mathrm{T}}}\right)^{[3]}\mathbf{X}_{\mathrm{p}}^{[3]}\right]=\left(\frac{\partial g_{\mathrm{d}}(\mathbf{X})}{\partial\mathbf{X}^{\mathrm{T}}}\right)^{[3]}\mathbf{C}_3(\mathbf{X}) \tag{10b}$$

$$\eta_{\mathrm{g}}=\mathrm{C}_4[g(\mathbf{X})]=\varepsilon^4\mathrm{E}\left[\left(\frac{\partial g_{\mathrm{d}}(\mathbf{X})}{\partial\mathbf{X}^{\mathrm{T}}}\right)^{[4]}\mathbf{X}_{\mathrm{p}}^{[4]}\right]=\left(\frac{\partial g_{\mathrm{d}}(\mathbf{X})}{\partial\mathbf{X}^{\mathrm{T}}}\right)^{[4]}\mathbf{C}_4(\mathbf{X}) \tag{10c}$$

where $\mathrm{Var}(\boldsymbol{X})$ is the variance matrix that includes all of the variance and covariance of the random parameters, $\mathbf{C}_3(\boldsymbol{X})$ and $\mathbf{C}_4(\boldsymbol{X})$ are the third and the fourth central moments matrices that include both of the third and the fourth central moments of the random parameters respectively. σ_{g}^2, θ_{g} and η_{g} are the variance and the third and fourth central moments of the state function $g(\boldsymbol{X})$ respectively.

The reliability index [15] is defined as

$$\beta=\frac{\mu_{\mathrm{g}}}{\sigma_{\mathrm{g}}}=\frac{\mathrm{E}[g(\mathbf{X})]}{\sqrt{\mathrm{Var}[g(\mathbf{X})]}} \tag{11}$$

It should be emphasized that the first-order approximations of μ_g and σ_g derived above must be evaluated at the mean values ($\mu_{X_1}, \mu_{X_2}, \cdots, \mu_{X_n}$). In some approximate sense, the reliability index may be directly used as a measure of reliability. If the distributions of the original random variables are normal, the distance from the "minimum" tangent plane to the failure surface may be used to approximate the actual failure surface, and the corresponding reliability, namely reliability, may be represented as

$$R = \Phi(\beta) \tag{12}$$

where $\Phi(\,)$ is the standard normal distribution function.

Because the state function $g(\boldsymbol{X})$ has non-normal distributions, Eq. (12) is not valid. In actual design, due to the lack of statistical dates, the probability density function or cumulative distribution function of some original random variables are often unknown, and the probabilistic characteristic of these variables is often expressed using only the first fourth moments. On the condition of known first four moments of original random variables, the probability distribution function of the standardized variable is approximately expressed by the first four moments of the original random variables, using the Edgeworth series.

3. Edgeworth series

For a state function $g(\boldsymbol{X})$, the standard forms can be expressed as

$$y = \frac{g(\mathbf{X}) - \mu_g}{\sigma_g} \tag{13}$$

The arbitrary distribution function of the standard random variable y that is approximately expressed by the standard normal distribution function using the Edgeworth series is addressed in [16]:

$$F(y) = \Phi(y) - \frac{\beta_1}{3!}\Phi^{(3)}(y) + \frac{\beta_2 - 3}{4!}\Phi^{(4)}(y) + \frac{10\beta_1^2}{6!}\Phi^{(6)}(y) \tag{14}$$

where the first four terms are available, and β_1, β_2 are the coefficient of skewness and the coefficient of kurtosis, respectively. $\Phi^{(i)}(\,)$ denotes the ith differentiation of $\Phi(\,)$:

$$\Phi^{(i)}(y) = (-1)^{i-1} H_{i-1}(y)\varphi(y) \tag{15}$$

where $\varphi(\,)$ is the standard normal probability density function and $H_{j-1}(\,)$ is the Hermite polynomial

$$\begin{cases} H_{j+1}(y) = yH_j(y) - jH_{j-1}(y) \\ H_0(y) = 1,\ H_1(y) = y \end{cases} \tag{16}$$

Thus, the reliability R is represented as

$$R(\beta) = P(g(\mathbf{X}) \geq 0) = 1 - F(-\beta) \tag{17}$$

When only the first four terms of the Edgeworth series are used, sometimes the reliability R>1 can occur when Eq. (17) is used to determine R. In the present study, if R>1 appeared, the amendatory expression from [4] was employed:

$$R^*(\beta) = R(\beta) - \left\{ \frac{R(\beta) - \Phi(\beta)}{\left\{1 + \left[R(\beta) - \Phi(\beta)\right]\beta\right\}^{\beta}} \right\} \tag{18}$$

According to reliability theory, the reliability R is between 0 and 1, namely, $0 \leq R \leq 1$. The amendatory expression (18) can ensure the reliability R to satisfy $0 \leq R \leq 1$ gradually and accurately.

4. Reliability sensitivity

The reliability index β is a function of the mean and standard deviation of $g(\boldsymbol{X})$ and also of $\boldsymbol{X}$. the reliability sensitivity with respect to the mean value of the system parameters is approximately derived as follows:

$$\frac{\mathrm{D}R(\beta)}{\mathrm{D}\bar{X}^{\mathrm{T}}} = \left(\frac{\partial R(\beta)}{\partial \beta} \cdot \frac{\partial \beta}{\partial \mu_g} \cdot \frac{\partial \mu_g}{\partial \bar{X}^{\mathrm{T}}}\right) + \left(\frac{\partial R(\beta)}{\partial \beta} \cdot \frac{\partial \beta}{\partial \sigma_g} \cdot \frac{\partial \sigma_g}{\partial \bar{X}^{\mathrm{T}}}\right) \tag{19}$$

Based on Eq. (6) and Eq. (10a),

$$\sigma_g^2 = \left(\frac{\partial g(\bar{\mathbf{X}})}{\partial \mathbf{X}^{\mathrm{T}}}\right)^{[2]} \mathrm{Var}(\mathbf{X}) = \left[\frac{\partial g(\bar{\mathbf{X}})}{\partial \mathbf{X}^{\mathrm{T}}} \otimes \frac{\partial g(\bar{\mathbf{X}})}{\partial \mathbf{X}^{\mathrm{T}}}\right] \mathrm{Var}(\mathbf{X}) = \mathbf{AB} \tag{20}$$

$$\frac{\partial \sigma_g}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} = \frac{1}{2\sigma_g} \cdot \frac{\partial \mathbf{AB}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} = \frac{1}{2\sigma_g} \cdot \left(\frac{\partial \mathbf{A}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}}(\mathbf{I}_n \otimes \mathbf{B}) + (\mathbf{I} \otimes \mathbf{A})\frac{\partial \mathbf{B}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}}\right) \tag{21}$$

where

$$\frac{\partial \mathbf{B}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} = \frac{\partial \mathrm{Var}(\mathbf{X})}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} = 0 \tag{22}$$

$$\frac{\partial \mathbf{A}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} = \frac{\partial^2 \bar{g}}{\partial^2 \bar{\mathbf{X}}^{\mathrm{T}}} \otimes \frac{\partial \bar{g}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} + (\mathbf{I} \otimes U_{1\times 1})\left(\frac{\partial^2 \bar{g}}{\partial^2 \bar{\mathbf{X}}^{\mathrm{T}}} \otimes \frac{\partial \bar{g}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}}\right)(\mathbf{I}_n \otimes U_{n\times n}) \tag{23}$$

where $\mathbf{I}_n$ is a unit matrix of dimensions $n\times n$ and $\mathbf{U}_{n\times n}$ is a transformation matrix of dimensions $n^2\times n^2$.

So,

$$\frac{\partial \sigma_g}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} = \frac{1}{2\sigma_g} \cdot \left(\frac{\partial^2 \bar{g}}{\partial^2 \bar{\mathbf{X}}^{\mathrm{T}}} \otimes \frac{\partial \bar{g}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} + (\mathbf{I} \otimes U_{1\times 1})\left(\frac{\partial^2 \bar{g}}{\partial^2 \bar{\mathbf{X}}^{\mathrm{T}}} \otimes \frac{\partial \bar{g}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}}\right)(\mathbf{I}_n \otimes U_{n\times n})\right)(\mathbf{I}_n \otimes \mathrm{Var}(\mathbf{X})) \tag{24}$$

In general, $\frac{\partial^2 \bar{g}}{\partial^2 \bar{\mathbf{X}}^{\mathrm{T}}}$ and $\mathrm{Var}(\mathbf{X})$ are very small, so $\frac{\partial \sigma_{\mathrm{g}}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}}$, also, is still very small. In (19), the second parameter of the reliability sensitivity with respect to the mean value is less than the first one, so the second parameter of the reliability sensitivity with respect to the mean value can be approximatively expressed as (25):

$$\frac{\mathrm{D}R(\beta)}{\mathrm{D}\bar{\mathbf{X}}^{\mathrm{T}}} = \frac{\partial R(\beta)}{\partial \beta}\frac{\partial \beta}{\partial \mu_{\mathrm{g}}}\frac{\partial \mu_{\mathrm{g}}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} \tag{25}$$

where

$$\frac{\partial R(\beta)}{\partial \beta} = \varphi(-\beta)\left\{1-\beta\left[\frac{1}{6}\frac{\theta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^3}H_2(-\beta)+\frac{1}{24}\left(\frac{\eta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^4}-3\right)H_3(-\beta) + \frac{1}{72}\left(\frac{\theta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^3}\right)^2 H_5(-\beta)\right] - \left[\frac{1}{3}\frac{\theta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^3}H_1(-\beta)+\frac{1}{8}\left(\frac{\eta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^4}-3\right)H_2(-\beta) + \frac{1}{8}\left(\frac{\eta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^4}-3\right)H_2(-\beta)+\frac{5}{72}\left(\frac{\theta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^3}\right)^2 H_4(-\beta)\right]\right\} \tag{26}$$

$$\frac{\partial \beta}{\partial \mu_{\mathrm{g}}} = \frac{1}{\sigma_{\mathrm{g}}} \tag{27}$$

$$\frac{\partial \mu_{\mathrm{g}}}{\partial \bar{\mathbf{X}}^{\mathrm{T}}} = \left[\frac{\partial \bar{g}}{\partial X_1}\ \frac{\partial \bar{g}}{\partial X_2}\ \cdots \frac{\partial \bar{g}}{\partial X_n}\right] \tag{28}$$

The reliability sensitivity with respect to the standard variance of the system parameters is approximately derived as follows:

$$\frac{\mathrm{D}R(\beta)}{\mathrm{DVar}(\mathbf{X})} = \left[\frac{\partial R(\beta)}{\partial \beta}\frac{\partial \beta}{\partial \sigma_{\mathrm{g}}}+\frac{\partial R(\beta)}{\partial \sigma_{\mathrm{g}}}\right]\frac{\partial \sigma_{\mathrm{g}}}{\partial \mathrm{Var}(\mathbf{X})} \tag{29}$$

where

$$\frac{\partial \beta}{\partial \sigma_{\mathrm{g}}} = -\frac{\mu_{\mathrm{g}}}{\sigma_{\mathrm{g}}^2} \tag{30}$$

$$\frac{\partial R(\beta)}{\partial \sigma_{\mathrm{g}}} = \varphi(-\beta)\left[\frac{1}{2}\frac{\theta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^4}H_2(-\beta)+\frac{1}{6}\frac{\eta_{\mathrm{g}}}{\sigma_{\mathrm{g}}^5}H_3(-\beta)+\frac{1}{12}\frac{\theta_{\mathrm{g}}^2}{\sigma_{\mathrm{g}}^7}H_5(-\beta)\right] \tag{31}$$

$$\frac{\partial \sigma_{\mathrm{g}}}{\partial \mathrm{Var}(\mathbf{X})} = \frac{1}{2\sigma_{\mathrm{g}}}\left[\frac{\partial \bar{g}}{\partial \mathbf{X}} \otimes \frac{\partial \bar{g}}{\partial \mathbf{X}}\right] \tag{32}$$

$$\frac{\partial \mu_{\mathrm{g}}}{\partial \mathrm{Var}(\mathbf{X})} = \frac{\partial g(\bar{\mathbf{X}})}{\partial \mathrm{Var}(\mathbf{X})} = 0 \tag{33}$$

Substituting the known conditions and the results derived earlier into Eqs. (25) and (29), the reliability sensitivity $\mathrm{D}R/\mathrm{D}\bar{\mathbf{X}}^{\mathrm{T}}$ and $\mathrm{D}R/\mathrm{DVar}(\mathbf{X})$ are obtained.

If the reliability computed by the Edgeworth series is R>1, the results computed by the amendatory expression (18) are closer to that by Monte Carlo simulation than that by the Edgeworth series in interval [0.99, 1] that is usually used for reliability analysis in engineering computation practice. The distribution function curves derived from the amendatory expression (18) are monotonic in interval [0, 1]. Therefore, if the reliability computed by the Edgeworth series is R>1, the reliability sensitivity computed by the differentiation of the amendatory expression is more accurate than that computed by Eqs. (25) and (29). (Sometimes the results computed by Eqs. (25) and (29) are erroneous). If the results computed by the Edgeworth series are R>1, the reliability sensitivity with respect to the reliability index is derived as follows:

$$\frac{\partial R^*(\beta)}{\partial \beta} = \frac{\partial R(\beta)}{\partial \beta} + \left[\frac{\partial R(\beta)}{\partial \beta} - \varphi(\beta)\right]\frac{\beta(\beta-1)[R(\beta)-\Phi(\beta)]-1}{\{1+[R(\beta)-\Phi(\beta)]\beta\}^{\beta+1}} + \frac{[R(\beta)-\Phi(\beta)](\{1+[R(\beta)-\Phi(\beta)]\beta\}\ln\{1+[R(\beta)-\Phi(\beta)]\beta\}+[R(\beta)-\Phi(\beta)]\beta)}{\{1+[R(\beta)-\Phi(\beta)]\beta\}^{\beta+1}} \tag{34}$$

Substituting Eq. (34) into $\partial R(\beta)/\partial \beta$ of Eqs. (25) and (29), the reliability sensitivity $\mathrm{D}R/\mathrm{D}\bar{\mathbf{X}}^{\mathrm{T}}$ and $\mathrm{D}R/\mathrm{DVar}(\mathbf{X})$ are obtained.

5. Numerical example

5.1 Reliability sensitivity of banjo flange

According to the Bach method [17], the bending stress on the banjo flange (in Fig. 1) risk section, namely at the D_1 diameter, using the structural model of an analogous beam is expressed as

$$\sigma = \frac{3P(D_0 - D_1)}{\pi D_1 h^2} \tag{35}$$

where P is total load effect on the flange, D_0 is the diameter of the bolt distributing circularity, D_1 is the diameter of the risk section, and h is the thickness of the flange.

On the basis of stress-strength interference theory, the state equation of the banjo flange is defined as

$$g(\mathbf{X}) = r - \sigma \tag{36}$$

where r is the material strength of the Banjo flange. The origi-

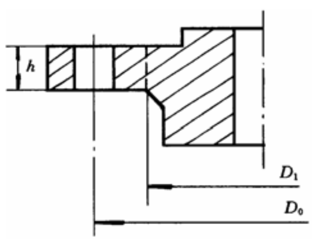

Fig. 1. Banjo flange.

nal random variable vector $\boldsymbol{X}$ is given by $\boldsymbol{X}=(r\ D_0\ D_1\ h\ P)^{\mathrm{T}}$, where the mean matrix $\mathrm{E}(\boldsymbol{X})$, the variance matrix $\mathrm{Var}(\boldsymbol{X})$, the third central moment matrix $\mathbf{C}_3(\boldsymbol{X})$ and the fourth central moment matrix $\mathbf{C}_4(\boldsymbol{X})$ are known.

Generally speaking, geometric parameters and material properties are thought to usually have normal distributions. The mean value and standard deviation of the geometric sizes of a Banjo flange are $(D_0)=(1200.0,\ 6.0)$ mm, $(D_1)=(1000.0,\ 5.0)$ mm, $(h)=(50.0,\ 0.25)$ mm. The total load P effecting the flange is an arbitrarily distributed random variable with the first four moments $(P)=(1.3025\times10^6\ \mathrm{N},\ 1.2021\times10^5\ \mathrm{N},\ 1.9872\times10^{15}\ \mathrm{N}^3,\ 1.1347\times10^{21}\ \mathrm{N}^4)$. The mean value and standard deviation of the material yield strength of the Banjo flange is $(r)=(135.0,\ 5.265)$ MPa. If there are fatigue problems, the strength value might be a fatigue-limit value according to the demand of life.

The reliability index β, reliability R and reliability sensitivities $\mathrm{D}R/\mathrm{D}\bar{\mathbf{X}}^{\mathrm{T}}$, $\mathrm{D}R/\mathrm{D}\mathrm{Var}(\mathbf{X})$, therefore, becomes

$\beta=3.100955$, $R_{\mathrm{E}}=0.9990356$, $R_{\mathrm{MCS}}=0.99349$,

$$\mathrm{D}R/\mathrm{D}\bar{\mathbf{X}}^{\mathrm{T}}=\begin{bmatrix}R_{(r)}\\R_{(D_0)}\\R_{(D_1)}\\R_{(h)}\\R_{(P)}\end{bmatrix}^{\mathrm{T}}=\begin{bmatrix}4.897\times10^{-4}\\-3.012\times10^{-4}\\3.639\times10^{-4}\\2.499\times10^{-3}\\-3.933\times10^{-8}\end{bmatrix}^{\mathrm{T}},$$

$$\mathrm{D}R/\mathrm{D}\mathrm{Var}(\mathbf{X})=\begin{bmatrix}R_{\mathrm{Var}(r)}\\R_{\mathrm{Cov}(r,D_0)}\\R_{\mathrm{Cov}(r,D_1)}\\R_{\mathrm{Cov}(r,h)}\\R_{\mathrm{Cov}(r,P)}\\R_{\mathrm{Cov}(D_0,r)}\\R_{\mathrm{Var}(D_0)}\\R_{\mathrm{Cov}(D_0,D_1)}\\R_{\mathrm{Cov}(D_0,h)}\\R_{\mathrm{Cov}(D_0,P)}\\R_{\mathrm{Cov}(D_1,r)}\\R_{\mathrm{Cov}(D_1,D_0)}\\R_{\mathrm{Var}(D_1)}\\R_{\mathrm{Cov}(D_1,h)}\\R_{\mathrm{Cov}(D_1,P)}\\R_{\mathrm{Cov}(h,r)}\\R_{\mathrm{Cov}(h,D_0)}\\R_{\mathrm{Cov}(h,D_1)}\\R_{\mathrm{Var}(h)}\\R_{\mathrm{Cov}(h,P)}\\R_{\mathrm{Cov}(P,r)}\\R_{\mathrm{Cov}(P,D_0)}\\R_{\mathrm{Cov}(P,D_1)}\\R_{\mathrm{Cov}(P,h)}\\R_{\mathrm{Var}(P)}\end{bmatrix}=\begin{bmatrix}-3.030\times10^{4}\\1.507\times10^{-4}\\-1.809\times10^{-4}\\-1.206\times10^{-3}\\2.315\times10^{-8}\\1.507\times10^{-4}\\-7.499\times10^{-5}\\8.999\times10^{-5}\\5.999\times10^{-4}\\-1.152\times10^{-8}\\-1.809\times10^{-4}\\8.999\times10^{-5}\\-1.080\times10^{-4}\\-7.199\times10^{-4}\\1.382\times10^{-8}\\-1.206\times10^{-3}\\5.999\times10^{-4}\\-7.199\times10^{-4}\\-4.800\times10^{-3}\\9.212\times10^{-8}\\2.315\times10^{-8}\\-1.152\times10^{-8}\\1.382\times10^{-8}\\9.212\times10^{-8}\\-1.768\times10^{-12}\end{bmatrix}$$

New insight into parametric sensitivity in reliability theory is given. The derivatives of the reliability with respect to the vector of random parameters $\boldsymbol{X}$ were established. As expected from this example, the reliability of the system increases as the material strength r, the diameter of risk section D_1 and the thickness h of the banjo flange increase, but the reliability descends as the total load P effect on the flange and the diameter of the bolt distributing circularity D_0 of the banjo flange rises. In other words, the results established that the reliability is very sensitive to the geometrical sizes, moderately sensitive to the material strength, and somewhat sensitive to the load of observation. The results of the reliability sensitivities largely accord with the practical operational conditions.

5.2 Reliability sensitivity of coil- tube spring

The most stress that the coil tube-spring [18] imparts to the walls of the vessel (in Fig. 2) is

$$S_{\max}=KS_{\mathrm{nom}} \tag{37}$$

where K is the shear stress factor and S_{nom} is the nominal shear stress of the coil tube-spring.

$$K=\frac{5}{4C}+\frac{7+3B^2}{8C^2} \tag{38}$$

$$C=\frac{D}{d},\quad B=\frac{d_1}{d} \tag{39}$$

where C is the index of the spring, B is the ratio of the outside diameter to the inside diameter, D is the pitch diameter of the spring, d is the outside diameter of the tube section and d_1 is the inside diameter of the tube section.

$$S_{\mathrm{nom}}=\frac{8PD}{\pi d^3\left(1-B^4\right)} \tag{40}$$

where P is the axial load.

As the strain factor equals 1, the stiffness of the coil tube-spring is expressed as

$$\frac{P}{\delta}=\frac{Gd^4\left(1-B^4\right)}{8D^3n} \tag{41}$$

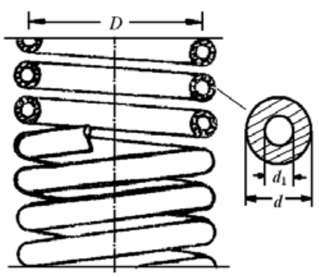

Fig. 2. Coil tube-spring.

where δ is the axial strain of the coil tube-spring, G is the shear modulus, and n is the number of working coils of the coil tube-spring.

The most shear stress is represented by substituting Eq. (38) and Eq. (41) into Eq. (37), as given by

$$S_{\max} = \left(\frac{5d}{4D} + \frac{7d^2 + 3d_1^2}{8D^2} \right) \frac{Gd}{\pi D^2 n} \delta \tag{42}$$

The influence of the load eccentricity and pitch effect is ignored in Eq. (42).

On the basis of the stress-strength interference theory, the state equation of the coil tube-spring is

$$g(\mathbf{X}) = r - S_{\max} \tag{43}$$

where r is the material strength of the coil tube-spring. The original random variable vector $\boldsymbol{X}$ is given by $\boldsymbol{X}=(r\ d_1\ d\ D\ G\ n\ \delta)^{\mathrm{T}}$, where the mean matrix E($\boldsymbol{X}$), the variance matrix Var($\boldsymbol{X}$), the third central moment matrix $\mathbf{C}_3(\boldsymbol{X})$ and the fourth central moment matrix $\mathbf{C}_4(\boldsymbol{X})$ are known.

The first two moments of the section sizes and material characteristics of a coil tube-spring are (d_1)=(9.4, 0.047) mm, (d)=(10.00076, 0.0500038) mm, (D)=(82.11195, 0.41055975) mm, (G)=(79380, 3969) MPa, (n)=(10, 0.0833), ρ=0.7. The material strength of the coil tube-spring r is determined by the fatigue limit. It is an arbitrarily distributed random variable with the first four moments (524.1855 MPa, 72.8453 MPa, -2.1176×10^5 (MPa)^3, 9.4475×10^7 (MPa)^4). The deformation quantity δ of the coil tube-spring is an arbitrary distributed random variable with the first four moments (δ)=(180.4157 mm, 3.5926 mm, 52.1762 (mm)^3, 888.3206 (mm)^4).

The reliability index β, reliability R and reliability sensitivities $\mathrm{D}R/\mathrm{D}\bar{\mathbf{X}}^{\mathrm{T}}$, $\mathrm{D}R/\mathrm{D}\mathrm{Var}(\mathbf{X})$ can be obtained as

β=4.634453, R_{E}=0.9999349, R_{MCS}=0.99974,

$$\mathrm{D}R/\mathrm{D}\mathbf{X}^{\mathrm{T}} = \begin{bmatrix} R_{\mathrm{E}(r)} \\ R_{\mathrm{E}(d_1)} \\ R_{\mathrm{E}(d)} \\ R_{\mathrm{E}(D)} \\ R_{\mathrm{E}(G)} \\ R_{\mathrm{E}(n)} \\ R_{\mathrm{E}(\delta)} \end{bmatrix}^T = \begin{bmatrix} 3.172\times10^{-6} \\ -2.730\times10^{-6} \\ -1.242\times10^{-4} \\ 2.273\times10^{-5} \\ -7.386\times10^{-9} \\ 9.738\times10^{-5} \\ -3.327\times10^{-6} \end{bmatrix}^{\mathrm{T}},$$

$$\mathrm{D}R/\mathrm{D}\mathrm{Var}(\mathbf{X}) = \begin{vmatrix} R_{\mathrm{Var}(r)} \\ R_{\mathrm{Cov}(r,d_1)} \\ R_{\mathrm{Cov}(r,d)} \\ R_{\mathrm{Cov}(r,D)} \\ R_{\mathrm{Cov}(r,G)} \\ R_{\mathrm{Cov}(r,n)} \\ R_{\mathrm{Cov}(r,\delta)} \\ R_{\mathrm{Cov}(d_1,r)} \\ R_{\mathrm{Var}(d_1)} \\ R_{\mathrm{Cov}(d_1,d)} \\ R_{\mathrm{Cov}(d_1,D)} \\ R_{\mathrm{Cov}(d_1,G)} \\ R_{\mathrm{Cov}(d_1,n)} \\ R_{\mathrm{Cov}(d_1,\delta)} \\ R_{\mathrm{Cov}(d,r)} \\ R_{\mathrm{Cov}(d,d_1)} \\ R_{\mathrm{Var}(d)} \\ R_{\mathrm{Cov}(d,D)} \\ R_{\mathrm{Cov}(d,G)} \\ R_{\mathrm{Cov}(d,n)} \\ R_{\mathrm{Cov}(d,\delta)} \\ R_{\mathrm{Cov}(D,r)} \\ R_{\mathrm{Cov}(D,d_1)} \\ R_{\mathrm{Cov}(D,d)} \\ R_{\mathrm{Var}(D)} \\ R_{\mathrm{Cov}(D,G)} \\ R_{\mathrm{Cov}(D,n)} \\ R_{\mathrm{Cov}(D,\delta)} \\ R_{\mathrm{Cov}(G,r)} \\ R_{\mathrm{Cov}(G,d_1)} \\ R_{\mathrm{Cov}(G,d)} \\ R_{\mathrm{Cov}(G,D)} \\ R_{\mathrm{Var}(G)} \\ R_{\mathrm{Cov}(G,n)} \\ R_{\mathrm{Cov}(G,\delta)} \\ R_{\mathrm{Cov}(n,r)} \\ R_{\mathrm{Cov}(n,d_1)} \\ R_{\mathrm{Cov}(n,d)} \\ R_{\mathrm{Cov}(n,D)} \\ R_{\mathrm{Cov}(n,G)} \\ R_{\mathrm{Var}(n)} \\ R_{\mathrm{Cov}(n,\delta)} \\ R_{\mathrm{Cov}(\delta,r)} \\ R_{\mathrm{Cov}(\delta,d_1)} \\ R_{\mathrm{Cov}(\delta,d)} \\ R_{\mathrm{Cov}(\delta,D)} \\ R_{\mathrm{Cov}(\delta,G)} \\ R_{\mathrm{Cov}(\delta,n)} \\ R_{\mathrm{Var}(\delta)} \end{vmatrix} = \begin{vmatrix} -3.799\times10^{-7} \\ 4.489\times10^{-7} \\ 6.783\times10^{-6} \\ -1.239\times10^{-6} \\ 4.145\times10^{-10} \\ -5.303\times10^{-6} \\ 1.824\times10^{-7} \\ 1.489\times10^{-7} \\ -1.232\times10^{-7} \\ -5.613\times10^{-6} \\ 1.025\times10^{-6} \\ -3.430\times10^{-10} \\ 4.388\times10^{-6} \\ -1.509\times10^{-7} \\ 6.783\times10^{-6} \\ -5.613\times10^{-6} \\ -2.558\times10^{-4} \\ 4.673\times10^{-5} \\ -1.563\times10^{-8} \\ 2.000\times10^{-4} \\ -6.877\times10^{-6} \\ -1.239\times10^{-6} \\ 1.025\times10^{-6} \\ 4.673\times10^{-5} \\ -8.537\times10^{-6} \\ 2.855\times10^{-9} \\ -3.653\times10^{-5} \\ 1.256\times10^{-6} \\ 4.145\times10^{-10} \\ -3.430\times10^{-10} \\ -1.563\times10^{-8} \\ 2.855\times10^{-9} \\ -9.550\times10^{-13} \\ 1.222\times10^{-8} \\ -4.202\times10^{-10} \\ -5.303\times10^{-6} \\ 4.388\times10^{-6} \\ 2.000\times10^{-4} \\ -3.653\times10^{-5} \\ 1.222\times10^{-8} \\ -1.563\times10^{-4} \\ 5.376\times10^{-6} \\ 1.824\times10^{-7} \\ -1.509\times10^{-7} \\ -6.877\times10^{-6} \\ 1.256\times10^{-6} \\ -4.202\times10^{-10} \\ 5.376\times10^{-6} \\ -1.849\times10^{-7} \end{vmatrix}$$

The derivatives of the reliability with respect to the vector of random parameters $\boldsymbol{X}$ are established. As expected from this example, the reliability of the system increases as the material strength r, the pitch diameter D and the number of working coils n increases, but the reliability diminishes as the outside diameter d, the inside diameter d_1, the shear modulus G and the axial strain δ of the coil tube-spring rise. In other words, the results indicated that the reliability is very sensitive to the geometrical sizes, moderately sensitive to the material strength and the axial strain, and somewhat sensitive to the shear modulus of observation.

6. Conclusions

This paper investigates a computational method of calculating the reliability sensitivity of mechanical components with arbitrary distribution parameters. Using the approach, the reliability-based sensitivity of mechanical components was determined quantitatively. Based on the reliability theory and a sensitivity analysis, the reliability-based sensitivity of mechanical components could be successfully derived, which is an important contribution to the field of mechanical components reliability research. The numerical computational results in this paper largely accord with the practical operational conditions. This method is effective, reliable and represents a potential theoretic basis for reliability-based design of mechanical components. Similarly, if case studies are prohibitively complicated, the implicit limit-state functions, such as those defined by the large-scale finite element models or the response surface method models, should be used to demonstrate the general applicability of the proposed method.

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